Signals and Systems

Sorting and testing systems

The notes for the lesson “Sorting and testing systems”: the chapters “Classification of systems: CT and DT systems” and “System properties: linearity, time invariance, causality, stability”, with their figures, worked examples, exam traps, quizzes and exam question bank, in one file.

2 chapters

About these notes

These are the notes for the University free n easy video lesson **Sorting and testing systems** on Signals and Systems: the chapters “Classification of systems: CT and DT systems” and “System properties: linearity, time invariance, causality, stability”, with every figure, worked example, exam trap and quiz, and the long-answer exam questions on these chapters.

Classification of systems: CT and DT systems

A system is an operator that turns an input signal into an output signal. Write it as y(t)=T{x(t)} in continuous time and y[n]=T{x[n]} in discrete time. The same idea covers an RC filter, a car suspension and a monthly savings balance. Engineers classify systems by memory, causality, stability, linearity, time invariance and invertibility because each label predicts how the system will behave in analysis and design. This chapter sets out those labels and shows how blocks connect in series and parallel.

You need: The chapter “Signals and systems defined; CT and DT signals” for CT and DT signals, and The chapter “Standard signals: impulse, step, ramp, exponential” for the unit step used in the example.

A system maps input to output

A system is any rule, device or process that acts on an input signal and produces an output signal. Write the action as an operator T. For a continuous-time (CT) input x(t), the output is y(t)=T{x(t)}. For a discrete-time (DT) input x[n], the output is y[n]=T{x[n]}. Here t is real time in

seconds; n is an integer sample index with no unit. The set of allowed inputs is the input set, and the set of resulting outputs is the output set.

The label CT or DT comes from the independent variable of the signals, not from the hardware. An RC low-pass circuit with voltage $v_i(t)$ in and $v_o(t)$ out is CT. A car suspension with road height as input and body height as output is CT. The on/off command sequence for a room heater, recorded once per minute, is DT. A bank balance updated once per month, with balance B[n], deposits d[n] and interest rate r, is DT: $B[n]=B[n-1]+d[n]+rB[n-1]$.

A computer can simulate a CT model by taking small time steps. The model is still CT because its signals are defined for every real t. This distinction matters when you sample a CT signal for processing; see Sampling theorem and reconstruction of signals.

Series and parallel connection

Large systems are built from smaller blocks. Two connections appear everywhere. In a series or cascade connection, the output of the first system feeds the second: $y(t)=T_2\{T_1\{x(t)\}\}$. In a parallel connection, the same input drives both blocks and the outputs add: $y(t)=T_1\{x(t)\}+T_2\{x(t)\}$.

The DT forms are identical with square brackets: $y[n]=T_2\{T_1\{x[n]\}\}$ for series, and $y[n]=T_1\{x[n]\}+T_2\{x[n]\}$ for parallel. The order matters in series. A filter followed by an amplifier is not the same as the amplifier followed by the filter when the amplifier clips.

An audio chain is a series connection: microphone, equaliser, amplifier, speaker. A stereo mix is a parallel connection: left and right channels add into a sum. In a control loop the blocks also connect in feedback, but that needs transfer-function tools (see Laplace transform analysis).

When blocks are linear and time-invariant, let $h_1(t)$ and $h_2(t)$ be their impulse responses. A series connection has overall impulse response $h_1(t)*h_2(t)$, a convolution; in DT use $h_1[n]*h_2[n]$. A parallel connection has $h_1(t)+h_2(t)$, or $h_1[n]+h_2[n]$ in DT. In the Laplace domain these become the product $H_1(s)H_2(s)$ and the sum $H_1(s)+H_2(s)$; in the z domain use $H_1(z)H_2(z)$ and $H_1(z)+H_2(z)$. Convolution itself is developed under Convolution integral and its evaluation.

Six labels used to classify systems

Classification answers practical questions before any calculation. The labels below are independent: a system can be, for example, causal and nonlinear.

- Memoryless (static) or with memory (dynamic). A memoryless system uses only the present input. $y(t)=3x(t)+2$ is memoryless. A system with memory uses past or future inputs, or past outputs. $y[n]=x[n]-x[n-1]$ has memory because it needs $x[n-1]$.

- Causal or noncausal. A causal system output at any time depends only on present and past inputs, never future inputs. $y[n]=x[n]+x[n+1]$ is noncausal because it looks one step ahead. All real-time physical systems are causal; stored data can be processed noncausally.
- Stable or unstable. A system is bounded-input bounded-output (BIBO) stable if every bounded input gives a bounded output. An audio amplifier with a fixed gain is stable; a microphone too close to its own speaker can become unstable.
- Linear or nonlinear. A system is linear if it obeys superposition: $T\{a\,x_1+b\,x_2\}=a\,T\{x_1\}+b\,T\{x_2\}$ for any inputs x_1 and x_2 and constants a and b. $y(t)=3x(t)+2$ is nonlinear because of the +2. $y[n]=x[n]-x[n-1]$ is linear.
- Time-invariant or time-varying. A time-invariant system does not change its rule with time: a shift in the input produces the same shift in the output. A resistor is time-invariant; a circuit switched on at t=0 may be time-varying during start-up.
- Invertible or noninvertible. A system is invertible if different inputs always give different outputs, so an inverse system can recover the input. $y(t)=3x(t)$ is invertible with inverse $x(t)=y(t)/3$. $y[n]=x[n]^2$ is not invertible over all real inputs because x and −x give the same output.

The formal tests for linearity, time invariance, causality and stability are in The chapter “System properties: linearity, time invariance, causality, stability”. Here the aim is the meaning, not the test routine. Some syllabi call this system classification and speak of a system as an interconnection of elements.

Reading a system equation

To classify a system from its equation, ask three questions in order. First, is the independent variable t or n? That fixes CT or DT. Second, does the output at an index use any other index? If yes, the system has memory; if no, it is memoryless. Third, does the output use a future index? If yes, it is noncausal.

1. Write the output for a general input, not for one numerical input.
2. List every input value that appears on the right-hand side.
3. If only plain x(t) or plain x[n] appears, the system is memoryless. Any other argument means memory: a delay x(t−a) or x[n−k], an advance x(t+a) or x[n+k], or a scale x(2t) or x[2n] (a, k > 0).
4. An advance makes the system noncausal, and so does a scale such as x(2t), because at t = 1 s it needs x(2), a future value. A delay alone keeps the system causal.
5. Check any added constant or product of inputs: either can make the system nonlinear.

A CT model is used for a physical device described by differential equations. A DT model is used for software, sampled data and sequences. A single system, such as a digitally controlled motor,

may need both: a CT model of the motor and a DT controller. Sampling links the two.

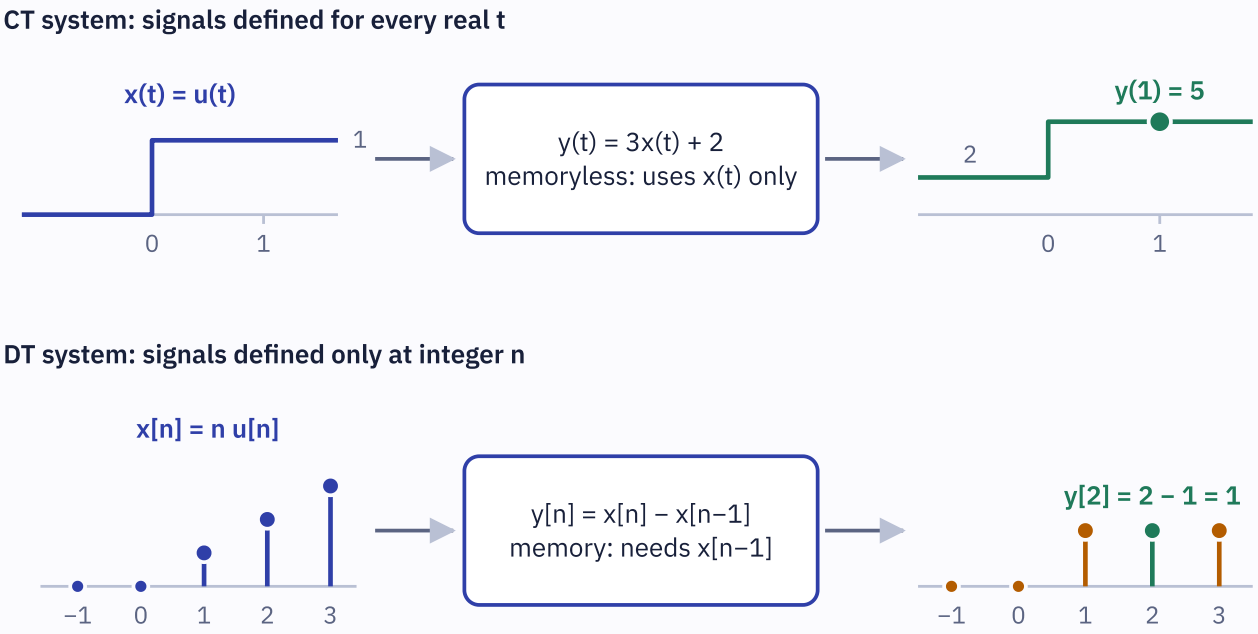

Figure 1 The two systems of the worked example with their inputs and outputs. The CT system only rescales and lifts the present value, so y(1) = 5; the DT system subtracts the previous sample, so the ramp n u[n] becomes 0, 1, 1, 1 and y[2] = 1.

WORKED EXAMPLE

Question. A CT system is $y(t)=3x(t)+2$ and a DT system is $y[n]=x[n]-x[n-1]$. Classify each as CT or DT, and as memoryless or with memory. For $x(t)=u(t)$, find $y(1)$; for $x[n]=n\,u[n]$, find $y[2]$. Justify each label in one line.

1. Read the independent variable. The first system uses t, so it is CT. The second uses n, so it is DT.
2. Check memory. In $y(t)=3x(t)+2$ the output uses only $x(t)$, so the CT system is memoryless. In $y[n]=x[n]-x[n-1]$ the output uses $x[n-1]$, so the DT system has memory.
3. For the CT input $x(t)=u(t)$, at $t=1$ the unit step value is $x(1)=u(1)=1$ because 1 is positive.
4. Substitute: $y(1)=3\times1+2=5$.
5. For the DT input $x[n]=n\,u[n]$, $x[2]=2u[2]=2\times1=2$ and $x[1]=1u[1]=1\times1=1$.
6. Substitute: $y[2]=x[2]-x[1]=2-1=1$.

Answer. The CT system is memoryless with $y(1)=5$, and the DT system has memory with $y[2]=1$, in the units of the signals.

EXAM TRAPS

- CT and DT describe the independent variable. A digital computer that simulates a CT model does not make the model DT.
- Causal is not the same as memoryless. $y[n]=x[n-1]$ is causal but has memory; $y[n]=x[n+1]$ is noncausal.
- A constant term breaks linearity. $y(t)=3x(t)+2$ is nonlinear and time-invariant, not linear.
- Stability and time invariance are independent. A system can be time-invariant and unstable, for example $y[n]=2y[n-1]+x[n]$, a recursive system whose feedback gain 2 doubles the output every step: for $x[n] = \delta[n]$ it gives $y[n] = 2^n u[n]$, which grows without bound.
- Squaring is not a small detail: $y[n]=x[n]^2$ is nonlinear over real inputs and not invertible, because x and $-x$ give the same output.

CHECK YOURSELF

– REMEMBER **Write the operator forms for a CT system and a DT system.** 2 marks

$y(t)=T\{x(t)\}$ for CT; $y[n]=T\{x[n]\}$ for DT; t is real time, n is an integer index.

– UNDERSTAND **Why does a numerical simulation not turn a CT model into a DT system?** 3 marks

The model's signals are still defined for every real t; the simulation only computes samples at small steps.

– APPLY **For $y[n]=x[n]+0.5x[n-1]$, find $y[2]$ when $x[0]=4$, $x[1]=2$, $x[2]=6$.** 4 marks

$y[2]=6+0.5\times2=6+1=7$.

– ANALYZE **Is $y(t)=x(t-1)+3$ time-invariant? Give a one-line reason.** 4 marks

Yes. Shifting x by t_0 shifts y by t_0; the $+3$ is unchanged.

– REMEMBER **State the BIBO stability condition.** 2 marks

Every bounded input produces a bounded output.

– UNDERSTAND **Give one CT and one DT physical example.** 3 marks

An RC circuit is CT; a monthly bank balance is DT.

– APPLY **For y(t)=3x(t)+2 and x(t)=u(t), find y(−1).** 4 marks

u(−1)=0, so y(−1)=3×0+2=2.

– ANALYZE **Is y[n]=x[n]² over all real inputs invertible? Explain.** 4 marks

No. x[n]=2 and x[n]=−2 both give y[n]=4, so the input cannot be recovered.

BEYOND THE SYLLABUS

Not for the exam. System classification feeds model-based design in control, communications and instrumentation. Digital twins keep a CT model of a plant beside a DT controller and update both from sensor data. In machine learning, a layer is a system with memory only through its stored state, so causality and stability become training constraints. Search for 'hybrid system', 'digital twin' and 'real-time simulation' to go further.

System properties: linearity, time invariance, causality, stability

A system is judged by four separate questions. Does it obey superposition (linearity)? Does a delayed input give only a delayed output (time invariance)? Does the output at an instant depend only on past and present input (causality)? Does every bounded input give a bounded output (BIBO stability)? Each question has a short definition and is answered with a counterexample when the system fails. These four tests decide whether the LTI tools of convolution and the Laplace transform apply, so they are asked early in every signals and systems exam.

You need the system operator idea and CT/DT notation from The chapter “Signals and systems defined; CT and DT signals”, and the impulse response symbol h(t) or h[n] from The chapter “Standard signals: impulse, step, ramp, exponential”.

Linearity: additivity plus scaling

Write the system as y = T{x}, where T is the operator, x is the input signal and y is the output signal. The system is linear when two inputs together give two outputs together, with constants kept outside.

$$T\{a\,x_1 + b\,x_2\} = a\,T\{x_1\} + b\,T\{x_2\}$$

x_1 and x_2 are any two input signals; a and b are any constants (real or complex); $T\{x_1\}$ and $T\{x_2\}$ are the outputs for those inputs.

The formula packs two rules. Set a = b = 1 to get additivity: $T\{x_1 + x_2\} = T\{x_1\} + T\{x_2\}$. Set b = 0 to get scaling (homogeneity): $T\{a\,x_1\} = a\,T\{x_1\}$. A system must pass both to be linear.

Linearity exists so that a complicated input can be split into simple pieces, each piece pushed through the system, and the outputs added. The impulse, step and exponential of The chapter “Standard signals: impulse, step, ramp, exponential” are exactly such pieces.

A quick test: a linear system must give zero output for zero input, because $T\{0\} = T\{0 \cdot x\} = 0 \cdot T\{x\} = 0$. So $y(t) = x(t) + 2$ is not linear, while $y(t) = t\,x(t)$ is linear even though its coefficient changes with time.

Time invariance and causality

Time invariance is about shifts, not about the system ageing. Delay the input by t_0 seconds and the whole output must delay by the same t_0, unchanged in shape.

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x(t − t₀) → y(t − t₀)   for every t₀;   x[n − n₀] → y[n − n₀] for every integer n₀
```

t_0 is the delay in seconds for continuous time (CT); n_0 is the delay in samples for discrete time (DT); y is the output produced by x.

1. Take a test input x and find the output $y = T\{x\}$.
2. Form the shifted input $x(t - t_0)$ and compute $T\{x(t - t_0)\}$.
3. Shift the original output to get $y(t - t_0)$.
4. Compare the two results symbolically. Equal for all t_0 means time-invariant; one t_0 with unequal results is a counterexample.

Causality: the output at a chosen instant t_0 may depend only on input values at times $t \le t_0$, never on future input. A system that needs the next 5 ms of input is non-causal. Live systems such as a phone call or a motor controller must be causal. Recorded-data work, such as smoothing a stored ECG file, is allowed to be non-causal.

For an LTI system, causality and BIBO stability both reduce to tests on the impulse response h(t). Causality holds if and only if $h(t) = 0$ for $t < 0$ ($h[n] = 0$ for $n < 0$ in DT); the stability test is in the table under BIBO stability below. Convolution and these LTI tests are developed under Causality and BIBO stability of LTI systems.

BIBO stability

BIBO means bounded-input bounded-output. The system is BIBO stable when every input that never exceeds some finite size produces an output that also never exceeds some finite size.

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if |x(t)| ≤ Mᵢₙ for all t, then |y(t)| ≤ Mₒᵤₜ for all t
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M_{in} is a finite bound on the input magnitude; M_{out} is a finite bound on the output magnitude; the same statement with n in place of t covers DT.

For an LTI system the test moves onto h alone.

DOMAIN	BIBO STABLE IFF	CAUSAL IFF
CT	$\int \|h(t)\| \, dt < \infty$	$h(t) = 0$ for $t < 0$
DT	$\Sigma \|h[n]\| < \infty$	$h[n] = 0$ for $n < 0$

The absolute value matters. A bounded input can be shaped to match the sign of h at every instant, so the output can reach the full sum of magnitudes. If that sum is infinite, one bounded input drives the output to infinity. Example: h(t) = u(t) (a pure integrator) is causal but not stable, since $\int|h(t)|dt$ over all time is infinite.

The four tests are independent. A system can be linear and unstable, causal and time-varying, or stable and non-causal. Answer them one by one.

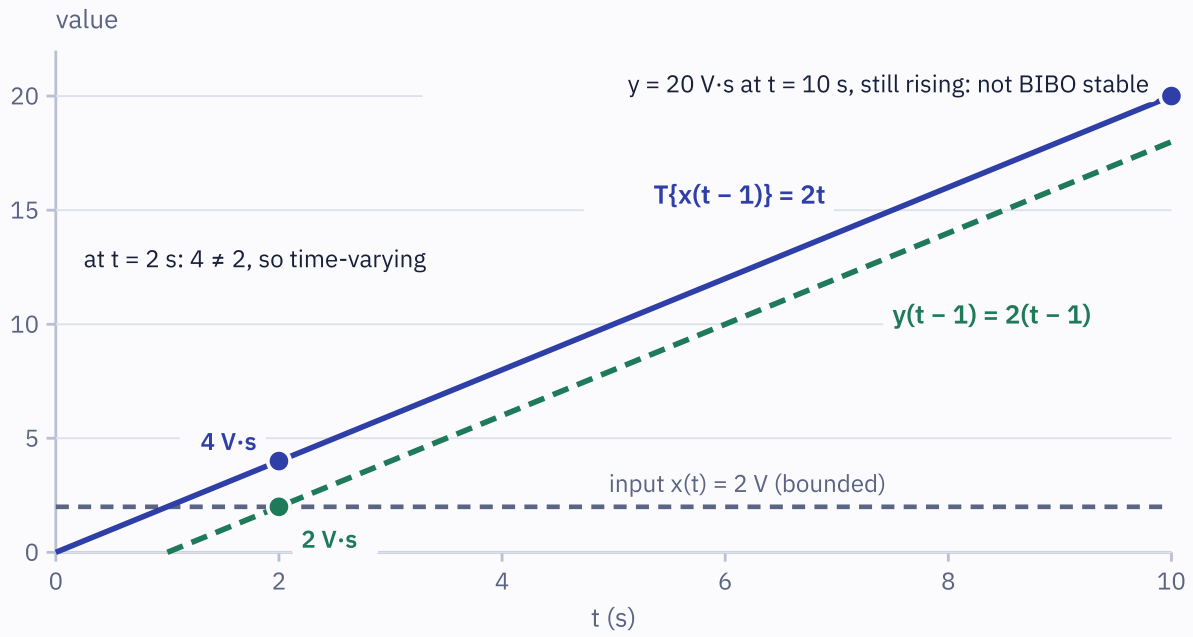

Figure 2 The two time-invariance results for y(t) = t x(t) with the constant input 2 V and delay 1 s. The delayed-input output 2t and the delayed output 2(t − 1) differ (4 against 2 V·s at t = 2 s), and the output keeps growing although the input stays at 2 V.

WORKED EXAMPLE

Question. A CT system is defined by $y(t) = t\,x(t)$, with t in seconds and y in volt-seconds when x is in volts. Test it for linearity, time invariance, causality and BIBO stability. Use $x(t) = 2$ V as the bounded test input and the delay $t_0 = 1$ s.

1. Linearity: $T\{a\,x_1 + b\,x_2\} = t(a\,x_1 + b\,x_2) = a{\cdot}t\,x_1 + b{\cdot}t\,x_2 = a\,T\{x_1\} + b\,T\{x_2\}$. Both additivity and scaling hold, so the system is linear.
2. Time invariance: the input $x(t - 1)$ gives the output $t\,x(t - 1)$. The shifted original output is $(t - 1)\,x(t - 1)$. Take $x = 2$ V and $t = 2$ s. First result: $2 \times 2 = 4$ V·s. Second result: $(2 - 1) \times 2 = 2$ V·s. They differ, so the system is time-varying.
3. Causality: the output at instant t uses $x(t)$ at that same instant only, never a future value. The system is causal.
4. Stability: take the bounded input $x(t) = 2$ V, so $M_{in} = 2$ V. Then $y(t) = 2t$ V·s. At $t = 10$ s, $|y| = 20$ V·s. As $t \to \infty$ the output grows without limit, so no finite M_{out} exists. The system is not BIBO stable.

Dry run

QUANTITY	EXPRESSION	VALUE
$t\,x(t - 1)$ at $t = 2$ s	2×2	4 V·s
$(t - 1)\,x(t - 1)$ at $t = 2$ s	1×2	2 V·s
y at $t = 10$ s	10×2	20 V·s

Answer. The system $y(t) = t\,x(t)$ is linear, time-varying, causal and not BIBO stable.

EXAM TRAPS

- Time invariance does not mean the system never changes with time. It means a shift in the input produces exactly that shift in the output. The coefficient t in t x(t) makes the system time-varying.
- A constant added term breaks linearity. For y(t) = x(t) + 2, the zero input gives 2, not 0, so the superposition formula fails no matter how small the constant is.
- Both linearity conditions are needed. Check scaling and additivity separately, because a system that fails either one is not linear. Example: $y(t) = x^2(t)$ fails scaling ($T\{a\,x\} = a^2x^2$, not $a\,x^2$) and fails additivity ($(x_1 + x_2)^2 \neq x_1^2 + x_2^2$).
- Causality is about which input samples are used, not about how fast the output appears. A system using x(t + 1) is non-causal even if its output is delayed by an hour.
- For LTI stability, take the absolute value inside the integral or sum. ∫h(t)dt finite does not guarantee stability; ∫|h(t)|dt must be finite.

CHECK YOURSELF

– REMEMBER **State the two conditions inside the linearity definition.** 2 marks

Additivity, $T\{x_1 + x_2\} = T\{x_1\} + T\{x_2\}$, and scaling (homogeneity), $T\{a\,x\} = a\,T\{x\}$ for any constant a.

– UNDERSTAND **Why does a linear system give zero output for zero input?** 2 marks

Set a = 0 in the scaling rule: $T\{0 \cdot x\} = 0 \cdot T\{x\} = 0$. So $T\{0\} = 0$, and any system with a nonzero zero-input output is not linear.

– APPLY **Test y[n] = x[2n] for time invariance. Use a counterexample with $n_0 = 1$.** 4 marks

Shifted input x[n – 1] gives x[2n – 1]. The shifted output is y[n – 1] = x[2(n – 1)] = x[2n – 2]. At n = 1 these are x[1] and x[0], which differ in general, so the system is time-varying. It is a decimator: it uses every second sample.

– APPLY **An LTI system has $h(t) = e^{-2t}\,u(t)$. Test it for causality and BIBO stability. Take the CT forms.** 4 marks

Causality: h(t) = 0 for t < 0 because of u(t), so it is causal. Stability: $\int_0^\infty e^{-2t}\,dt = 1/2 = 0.5$, which is finite, so the system is BIBO stable.

– ANALYZE **A system is causal and BIBO stable but not linear. Can it still be time-invariant? Give a one-line reason with an example.** 4 marks

Yes. The four properties are independent. Example: $y(t) = |x(t)|$ is causal (uses only present input), stable ($|y| = |x| \le M_{in}$), time-invariant (a shift in x gives the same shift in $|x|$), but not linear, since $|x_1 + x_2|$ is not always $|x_1| + |x_2|$.

– ANALYZE **Why does the stability test for an LTI system use $\int|h(t)|dt$ rather than $\int h(t)dt$?** 5 marks

A bounded input can be chosen to match the sign of h(t) at every instant, so every part of h adds with the same sign and the output can reach the full sum of magnitudes. If $\int|h(t)|dt$ is infinite, that bounded input produces an unbounded output. A finite $\int h(t)dt$ can still occur with signs cancelling, so it does not prove stability.

– UNDERSTAND **Give one practical situation where a non-causal system is acceptable.** 3 marks

Offline processing of a stored recording, such as smoothing a saved ECG or sharpening a photograph. The whole file is already available, so using future samples costs nothing and no real-time decision is delayed.

BEYOND THE SYLLABUS

Not for the exam. The four tests are the gate to the LTI toolbox: once a system is linear and time-invariant, convolution with h(t) describes it completely, and the Laplace and Z transforms turn it into algebra on poles and zeros. Stability tests then become region-of-convergence checks on the Laplace and Z transforms instead of integrals. In control and communications practice, engineers also measure gain margin and phase margin, and for nonlinear blocks they linearise around an operating point before applying these tests. Search for BIBO stability, impulse response absolute summability, and bounded-input bounded-output in control texts to go further.

Key terms of this lesson

TERM	MEANING OR FORMULA	CHAPTER
Memoryless system	The output at each instant uses only the input at that instant, such as $y(t) = 3x(t) + 2$.	“Classification of systems: CT and DT systems”
Linear system	A system is linear if it obeys superposition: $T\{a\,x_1+b\,x_2\}=a\,T\{x_1\}+b\,T\{x_2\}$ for all inputs and constants.	“System properties: linearity, time invariance, causality, stability”
Time-invariant system	A system is time-invariant if a delayed input gives the same output delayed by the same amount: $x(t-t_o)\rightarrow y(t-t_o)$.	“System properties: linearity, time invariance, causality, stability”
Causal system	A system is causal if its output at any time depends only on present and past inputs, not future inputs.	“System properties: linearity, time invariance, causality, stability”
BIBO stable system	A system is bounded-input bounded-output stable if every bounded input produces a bounded output.	“System properties: linearity, time invariance, causality, stability”

Exam question bank

Long-answer exam questions on these chapters, each with an outline of the points an examiner looks for. Answer one in full on paper before you read its outline.

Remember and understand

– REMEMBER **Define a continuous-time system and a discrete-time system. Give one electrical example of each and identify input and output.** 8 marks

Points to cover. CT system: input x(t) gives output y(t) for real t; DT system: input x[n] gives output y[n] for integer n; CT example: RC low-pass filter with voltage input and output; DT example: a digital filter acting on samples; State input and output clearly.

Model answer.

1. **CT system.** A system whose input and output are continuous-time signals: y(t) = T{x(t)}, with t real. It is usually described by a differential equation.
2. **DT system.** A system whose input and output are discrete-time signals: y[n] = T{x[n]}, with n an integer. It is usually described by a difference equation and runs on a processor.
3. **Electrical CT example: RC low-pass filter.** Input: the source voltage x(t) across the series R and C. Output: the capacitor voltage y(t). Kirchhoff's voltage law gives

```
RC dy/dt + y(t) = x(t)
```

4. With R = 1 kΩ and C = 1 µF, the time constant is RC = 1 ms. For a 1 V step input starting from rest, $y(t) = 1 - e^{-t/RC}$ V, so $y(1\text{ ms}) = 1 - e^{-1} = 0.632$ V. Both signals exist at every real t, so the system is CT.
5. **Electrical DT example: two-point averaging filter.** An ADC samples a voltage; a microcontroller computes

```
y[n] = 0.5 x[n] + 0.5 x[n − 1]
```

6. Input: the samples x[n] in volts. Output: the samples y[n] in volts, sent to a DAC. With x[n] = 0, 2, 4, 6 V for n = 0 to 3 and x[−1] = 0:

N	X[N] (V)	X[N − 1] (V)	Y[N] (V)
0	0	0	0
1	2	0	1
2	4	2	3
3	6	4	5

7. The label comes from the independent variable, not the hardware: the ADC before this filter is a hybrid CT-to-DT block, and the filter itself is DT.

8 CT and DT systems side by side:

FEATURE	RC LOW-PASS FILTER (CT)	TWO-POINT AVERAGE (DT)
Input	source voltage x(t)	samples x[n] from the ADC
Output	capacitor voltage y(t)	samples y[n] to the DAC
Model	RC dy/dt + y = x	y[n] = 0.5x[n] + 0.5x[n − 1]
Memory	yes: the capacitor stores charge	yes: one past sample is stored
Response to a 1 V step	$1 - e^{-t/RC}$: 0.632 V at t = RC	0.5 V at n = 0, then 1 V from n = 1

9 Check: y[2] = 0.5(4) + 0.5(2) = 3 V, the average of the two most recent samples.

Apply

– APPLY **Test the system y(t) = t x(t) for linearity, time invariance, causality, and BIBO stability. Show the steps.** 14 marks

Points to cover. Linearity: $T\{a\,x_1 + b\,x_2\} = t(a\,x_1 + b\,x_2) = a\,y_1 + b\,y_2$; Time invariance: input $x(t - t_0)$ gives $t\,x(t - t_0)$, not $(t - t_0)x(t - t_0)$; Causality: output at t uses input at the same t only; Stability: bounded $|x| \le M$ gives $|y| \le M|t|$, unbounded as $|t| \to \infty$; Conclusion: linear, time-varying, causal, not BIBO stable.

Model answer.

1. **Linearity.** Apply the input $a\,x_1(t) + b\,x_2(t)$:

```
T{a x₁ + b x₂} = t[a x₁(t) + b x₂(t)]
               = a·t x₁(t) + b·t x₂(t)
               = a y₁(t) + b y₂(t)
```

2. Additivity and scaling both hold for all inputs and constants, so the system is linear. It also gives zero output for zero input, as every linear system must.

3. **Time invariance.** Delay the input by t_0 and compare with the delayed output:

```
input x(t − t₀)        → output y₁(t) = t x(t − t₀)
delayed output         → y(t − t₀) = (t − t₀) x(t − t₀)
```

4. These differ whenever $t_0 \neq 0$. Numbers: $x(t) = 2$ V, $t_0 = 1$ s, $t = 2$ s. $y_1(2) = 2 \times 2 = 4$ V·s, but $y(2 - 1) = (2 - 1) \times 2 = 2$ V·s. One counterexample is enough: the system is time-varying. The coefficient t means the gain depends on when the input arrives.

5. **Causality.** The output at time t uses x(t) at that same instant only, never x at a later time. The system is causal (and memoryless).

6. **BIBO stability.** Take the bounded input $x(t) = 2$ V, so $|x(t)| \le M_{in} = 2$ V. Then $y(t) = 2t$ V·s, and $|y(t)| = 2|t|$ grows without limit as $|t| \to \infty$. No finite M_{out} bounds the output, so the system is not BIBO stable.

7. Trace of the numbers used:

QUANTITY	EXPRESSION	VALUE
$T\{x(t - 1)\}$ at t = 2 s	2 × 2	4 V·s
$y(t - 1)$ at t = 2 s	(2 − 1) × 2	2 V·s
y at t = 10 s	10 × 2	20 V·s
y at t = 100 s	100 × 2	200 V·s

8 Result.

PROPERTY	RESULT	REASON
Linear	yes	superposition holds for all inputs
Time-invariant	no	4 V·s ≠ 2 V·s at t = 2 s with t_0 = 1 s
Causal	yes	uses x at the present instant only
BIBO stable	no	input 2 V gives 2t, unbounded

9 A numeric linearity check: at t = 3 s, inputs x_1 = 1 V and x_2 = 2 V give y_1 = 3 and y_2 = 6 V·s; the input $x_1 + x_2$ = 3 V gives 3 × 3 = 9 V·s = $y_1 + y_2$, and the input $5x_1$ = 5 V gives 15 V·s = $5y_1$.

10 Why a constant input is a fair test of time invariance: a delayed constant is the same constant, so a time-invariant system would have to give the same output after the delay, shifted by 1 s. Here the output for the same input is 2t before and 2t after, but the shifted output should be 2(t − 1); the gain t depends on the clock, which is exactly what time-varying means.

11 A pulse input shows the same thing: x(t) = u(t) − u(t − 1) gives y = t on 0 to 1 s (peak 1 V·s), but the delayed pulse x(t − 1) gives y = t on 1 to 2 s (peak 2 V·s), a taller output, not a shifted copy.

12 Check: the properties are independent, so a linear causal system can still be time-varying and unstable. At t = 10 s, y = 10 s × 2 V = 20 V·s, as y = t·x says.

– APPLY **Test the discrete-time system y[n] = x[n] + x[n − 1] for linearity, time invariance, causality, and BIBO stability. Justify each answer.** 12 marks

Points to cover. Linearity follows from addition and scaling; Time invariance: a delayed input gives the same output delayed; Causality: uses present and past input only; Stability: |y[n]| ≤ |x[n]| + |x[n − 1]| ≤ 2M for bounded input; Conclusion: linear, time-invariant, causal, BIBO stable.

Model answer.

1. **Linearity.**

```
T{a x₁ + b x₂} = (a x₁[n] + b x₂[n]) + (a x₁[n − 1] + b x₂[n − 1])
               = a(x₁[n] + x₁[n − 1]) + b(x₂[n] + x₂[n − 1])
               = a y₁[n] + b y₂[n]
```

2. Superposition holds, so the system is linear.
3. **Time invariance.** Feed $x_1[n] = x[n - n_0]$:

```
T{x[n − n₀]} = x[n − n₀] + x[n − n₀ − 1]
y[n − n₀]    = x[n − n₀] + x[n − 1 − n₀]
```

4. The two are equal for every integer n_0, so the system is time-invariant. Its coefficients (1 and 1) do not depend on n.
5. Numeric trace with x[n] = 1, 2, 3 at n = 0, 1, 2 (0 elsewhere), and the same input delayed by $n_0 = 2$:

N	0	1	2	3	4	5
x[n]	1	2	3	0	0	0
y[n] = x[n] + x[n − 1]	1	3	5	3	0	0
x[n − 2]	0	0	1	2	3	0
T{x[n − 2]}	0	0	1	3	5	3
y[n − 2]	0	0	1	3	5	3

6. The last two rows match, as time invariance requires.
7. **Causality.** y[n] uses x[n] (present) and x[n − 1] (past), never x[n + 1]. The system is causal. It has memory, because it stores one past sample.
8. **BIBO stability.** If |x[n]| ≤ M for all n, then by the triangle inequality

```
|y[n]| ≤ |x[n]| + |x[n − 1]| ≤ M + M = 2M
```

9. Every bounded input gives an output bounded by 2M, so the system is BIBO stable. The LTI test agrees: the impulse response is h[n] = δ[n] + δ[n − 1], and Σ |h[n]| = 1 + 1 = 2, which is finite.

10. The system is LTI, so it is fully described by its impulse response: h[n] = δ[n] + δ[n − 1], the values 1, 1 at n = 0, 1. Every output is the convolution y[n] = x[n] * h[n]; for the trace above, 1, 2, 3 convolved with 1, 1 gives 1, 3, 5, 3, as found.

11. The bound 2M is reached: x[n] = M for all n gives y[n] = 2M. An alternating input x[n] = $M(-1)^n$ gives y[n] = $M(-1)^n + M(-1)^{n-1} = 0$, so this system passes slow inputs and blocks the fastest one, a simple low-pass filter.

12. Conclusion: linear, time-invariant, causal and BIBO stable. Check: with M = 3 in the trace above, the largest output is 5 ≤ 2 × 3 = 6.

Analyze and evaluate

– EVALUATE **Evaluate the statement: 'A causal system is always stable, and a linear system is always time-invariant.' Give definitions and counterexamples or reasoning.** 14 marks

Points to cover. Causality: output depends on present and past inputs only; Stability: every bounded input gives a bounded output; Counterexample causal but unstable: $y(t) = e^t x(t)$; Linearity: superposition holds; Counterexample linear but time-varying: $y(t) = t\,x(t)$; Conclusion: both claims are false; the properties are independent.

Model answer.

1. **Definitions.** Causal: y at time t depends only on x at times ≤ t. BIBO stable: $|x(t)| \le M_{in}$ for all t implies $|y(t)| \le M_{out}$ for all t, with M_{out} finite. Linear: $T\{a\,x_1 + b\,x_2\} = a\,T\{x_1\} + b\,T\{x_2\}$. Time-invariant: $x(t - t_0)$ gives $y(t - t_0)$ for every t_0.

2. **Claim 1: causal ⇒ stable. False.** Take $y(t) = e^t x(t)$. It uses only x(t), so it is causal. With the bounded input $x(t) = 1$, $y(t) = e^t$: $y(5) = 148.4$ and $y(10) = 22\,026$, with no bound. Causal but unstable.

3. A second counterexample, from LTI theory: the integrator $y(t) = \int x(\tau)\,d\tau$ over $\tau \le t$ has $h(t) = u(t)$. It is causal because $h(t) = 0$ for $t < 0$, but $\int |h(t)|\,dt = \infty$, and the input u(t) gives the ramp r(t), which grows without bound.

4. **Claim 2: linear ⇒ time-invariant. False.** Take $y(t) = t\,x(t)$. Superposition holds, so it is linear. With $x(t) = 2$ V and $t_0 = 1$ s, at $t = 2$ s: $T\{x(t - 1)\} = 2 \times 2 = 4$ V·s, but $y(t - 1) = (2 - 1) \times 2 = 2$ V·s. Linear but time-varying.

5. The converse claims fail too. $y(t) = x(t + 1)$ is stable ($|y| \le M_{in}$) but noncausal. $y(t) = x^2(t)$ is time-invariant but nonlinear: doubling the input quadruples the output.

6. Property table for the four systems:

SYSTEM	LINEAR	TIME-INVARIANT	CAUSAL	BIBO STABLE
$y(t) = e^t x(t)$	yes	no	yes	no
$y(t) = t\,x(t)$	yes	no	yes	no
$y(t) = x(t + 1)$	yes	yes	no	yes
$y(t) = x^2(t)$	no	yes	yes	yes

7. Check the last row: input $x = 3$ gives 9, input 6 gives $36 \ne 2 \times 9$, so x^2 is not linear; a delayed input gives a delayed square, so it is time-invariant; a bounded $|x| \le M$ gives $|y| \le M^2$.

8. For LTI systems the two properties are two different conditions on the impulse response: causal when $h(t) = 0$ for $t < 0$, stable when $\int |h(t)|\,dt$ is finite. $h(t) = e^{-2t}u(t)$ meets both ($\int |h|\,dt = 0.5$); $h(t) = u(t)$ is causal but $\int |h|\,dt = \infty$; $h(t) = e^{-2|t|}$ has $\int |h|\,dt = 0.5 + 0.5 = 1$, so it is stable, but it is nonzero for $t < 0$, so it is noncausal. Neither condition implies the other.

9. **Verdict.** Both parts of the statement are false. Linearity, time invariance, causality and stability are four independent questions; each must be tested from its own definition, and one counterexample is enough to fail a property.

Glossary (terms used in this lesson)

Causal system	A system whose output does not depend on future input values.
Independent variable	The quantity on which a signal depends, usually time t or integer index n.
Linear system	A system that obeys superposition for all inputs and constants.
Sample	Each value x[n] of a discrete-time signal; for a sampled signal it is $x(nT_s)$.
Signal	A function of an independent variable that carries information.
System	An operator that maps an input signal to an output signal.
Time-invariant system	A system whose response does not depend on the absolute time of the input.
Unit step	A signal that is zero before the origin and one from the origin on ($u(0) = 1$ in this book).

Sources and further reading

Textbooks

- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals and Systems*, 2nd edition: Chapter 1 covers signals, operations on the independent variable, standard signals and system properties.
- Simon Haykin, Barry Van Veen, *Signals and Systems*, 2nd edition: Chapter 1 covers signal and system basics, classifications, and operations.
- B. P. Lathi, *Linear Systems and Signals*, 2nd edition: Chapter 1 covers signal classification, operations, signal models and system classification; the Fourier series chapter treats signals as vectors and orthogonal sets.

Further reading online

- The Scientist and Engineer's Guide to Digital Signal Processing: www.dspguide.com
- TutorialsPoint: Signals and Systems: www.tutorialspoint.com/signals_and_systems/index.htm

Syllabus: the University free n easy universal syllabus, built from top universities' syllabi.

Built for the University free n easy YouTube channel from the universal syllabus of Signals and Systems (Oct 2026). Written with top universities' syllabi and standard textbooks; check a formula against your prescribed book before an exam, and report mistakes on the channel.